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An Overrelaxation Method for Euler Equations in Steady Transonic Flow

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The overrelaxation method is extended to Euler equations for steady transonic flow. The method has first order convergence rate. This is illustrated by one- and two-dimensional examples.

INTRODUCTION

In time marching methods for solving steady transonic Euler equations, the unsteady equations are integrated until steady state is reached. This procedure is suggested by the hyperbolic character of the system of equations with respect to time. The convergence rate of these methods, however, is extremely small. By defining convergence rate as the inverse of the number of iterations or time steps necessary to damp a perturbation of the steady state a fixed number of magnitudes, this convergence rate is found to be of the order of Δx^2. Relaxation methods for the steady potential equation have by the same definition a convergence rate of the order of Δx. Relaxation techniques, however, are not applicable to steady Euler equations since in the subsonic parts of the flow field the system of equations has simultaneously hyperbolic and elliptic features.

Overrelaxation to accelerate the convergence of unsteady Euler equations is possible since these are uniformally hyperbolic. First attempts to use overrelaxation are independently due to Wirz [1] and Désidéri and Tannehill [2]. The model equation,

$$(\partial\xi/\partial t) + V(\partial\xi/\partial x) = 0 \tag{1}$$

is discretized by Wirz as

$$\bar{\xi}(x, t+\Delta t) - \bar{\xi}(x, t) + (\lambda/2)(\xi(x+\Delta x, t) - \xi(x-\Delta x, t)) = 0 \tag{2}$$

$$\xi(x, t+\Delta t) = r_1\bar{\xi}(x, t+\Delta t) + r_2\bar{\xi}(x, t) + (1-r_1-r_2)\,\xi(x, t) \tag{3}$$

$\lambda = V\Delta t/\Delta x$, $\bar{\xi}$ is an artificial state variable. The discretisation of Désidéri and Tannehill on (1) has two steps.

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Downwind:

$$\begin{aligned} \xi(x, t+\tilde{\Delta t}) &= \xi(x, t) - \lambda(\xi(x+\Delta x, t) - \xi(x, t)), \\ \bar{\xi}(x, t+\Delta t) &= (1-r_1)\,\bar{\xi}(x, t) + r_1 \xi(x, t+\tilde{\Delta t}). \end{aligned} \tag{4}$$

Upwind:

$$\begin{aligned} \bar{\xi}(x, t+\tilde{\Delta t}) &= \bar{\xi}(x, t+\Delta t) - \lambda(\bar{\xi}(x, t+\Delta t) - \bar{\xi}(x-\Delta x, t+\Delta t)), \\ \xi(x, t+\Delta t) &= (1-r_2)\,\xi(x, t) + r_2 \bar{\xi}(x, t+\tilde{\Delta t}). \end{aligned} \tag{5}$$

Here again artificial variables are introduced. For both schemes, it can be shown that the convergence rate is of the order of Δx for an optimal choice of the relaxation factors [1, 2]. A drawback of both schemes is that the number of dependent variables is doubled by the use of the artificial state variables. This complicates the calculations since these artificial variables have dynamics. They need associated boundary conditions and they interact with the natural dependent variables.

In this article, an overrelaxation method is illustrated which avoids artificial state variables by the use of preliminary state variables. These preliminary variables are used in the same way as in the classic overrelaxation method for elliptic equations [3].

Definition of a Relaxation Method

We define a relaxation method as a method that fulfills three conditions:

(1) *The Order Condition.* In a relaxation method of order N, preliminary values $\bar{\xi}_1$, $\bar{\xi}_2, \ldots, \bar{\xi}_N$ are calculated for the state at level $t+\Delta t$. These are relaxed according to

$$\xi(t+\Delta t) = r_1\bar{\xi}_1 + r_2\bar{\xi}_2 + \cdots + r_N\bar{\xi}_N + (1 - r_1 - r_2 - \cdots - r_N)\,\xi(t).$$

(2) *The Serial Condition.* As soon as the state is calculated in some point (x, y, z) on the level $t+\Delta t$, the state at level t and the preliminary values at level t are no longer used.

(3) *The Convergence Rate Condition.* The asymptotic convergence rate is critically dependent upon the relaxation factors $r_1, r_2, \ldots, r_N$. In the vicinity of optimal relaxation, the asymptotic convergence rate is $O(\Delta x)$, elsewhere it is of a higher order.

According to this definition, the classic relaxation method for the potential equation is a first order method when the analogy between iteration levels and time levels is used. Through the serial condition, the character of the preliminary variables is different from that of the artificial variables in the previously cited methods. The preliminary variables need not be stored and do not introduce extra dynamics into the system.

In the sequel we shall prove that a method which fulfills these conditions can be constructed for time dependent equations of hyperbolic type, such as Euler equations. This generates a practical overrelaxation method for the solution of the steady from of these equations, even when the steady equations have a hybrid and mixed elliptic–hyperbolic character. The method is therefore applicable to steady transonic Euler equations.

A Relaxation Scheme for a One-Dimensional Transport Equation

Equation (1) is generally accepted to be a good linear model for Euler equations if V in (1) can have the characteristic velocities of Euler equations. In a one-dimensional case these are u, $u+c$, and $u-c$. Here u is the convective velocity and c is the velocity of sound. Hence in subsonic flow ($u<c$), V can have both positive and negative values. In supersonic flow ($u>c$), all characteristic velocities are positive. In transonic flow, the characteristic velocities can have a varying sign in the flow field.

First, we remark that it is impossible to construct a stable first order relaxation scheme for (1), when V can have both positive and negative values. A general first order scheme is

$$\xi(x, t+\widetilde{\Delta t}) = \xi(x, t) + \lambda(k\xi(x-\Delta x, t+\Delta t) + \alpha m\xi(x, t+\widetilde{\Delta t}) + \beta m\xi(x, t) + n\xi(x+\Delta x, t)) \tag{6}$$

with $k+m+n=0$, $\alpha+\beta=1$

$$\xi(x, t+\Delta t) = r\xi(x, t+\widetilde{\Delta t}) + (1-r)\,\xi(x, t).$$

This gives

$$(1-\alpha m\lambda)\,\xi(x, t+\Delta t) = rk\lambda\xi(x-\Delta x, t+\Delta t) + (1-\alpha m\lambda + rm\lambda)\,\xi(x, t) + rn\lambda\xi(x+\Delta x, t). \tag{7}$$

This scheme has the general form

$$D\xi(x, t+\Delta t) = A\xi(x-\Delta x, t+\Delta t) + B\xi(x, t) + C\xi(x+\Delta x, t) \tag{8}$$

with

$$D = 1-\alpha m\lambda, \qquad A = rk\lambda, \qquad B = 1-\alpha m\lambda + rm\lambda, \qquad C = rn\lambda.$$

The stability of scheme (8) can be studied by the substitution of a Fourier component

$$\xi(x, t) = \phi(t)\, e^{j\omega x}.$$

This gives the amplification factor

$$G = \phi(t + \Delta t)/\phi(t) = ((B + Ce^{j\phi})/(D - Ae^{-j\phi})) \tag{9}$$

with $\phi = \omega\, \Delta x$,

$$|G| < 1 \qquad \text{if} \quad (A + B)(A + C) \geqslant 0. \tag{10}$$

For the first order scheme, the term $A + C$ is proportional to λ, while $A + B$ contains a term independent of λ. Hence condition (10) cannot be fulfilled for both positive and negative values of λ. A stable relaxation scheme, therefore, should have at least second order. A general second order scheme is:

First Preliminary Value:

$$\begin{aligned}\xi(x, t + \widetilde{\Delta t}) = \xi(x, t) + \lambda(k_1 \xi(x - \Delta x, t + \Delta t) + \alpha_1 m_1 \xi(x, t + \widetilde{\Delta t})\\ + \beta_1 m_1 \xi(x, t) + n_1 \xi(x + \Delta x, t).\end{aligned} \tag{11}$$

Second Preliminary Value:

$$\begin{aligned}\xi(x, t + \widetilde{\widetilde{\Delta t}}) = \xi(x, t) + \lambda(k_2 \xi(x - \Delta x, t + \Delta t) + \alpha_2 m_2 \xi(x, t + \widetilde{\widetilde{\Delta t}})\\ + \beta_2 m_2 \xi(x, t + \widetilde{\Delta t}) + \gamma_2 m_2 \xi(x, t) + n_2 \xi(x + \Delta x, t)).\end{aligned} \tag{12}$$

Relaxation:

$$\xi(x, t + \Delta t) = r_1 \xi(x, t + \widetilde{\Delta t}) + r_2 \xi(x, t + \widetilde{\widetilde{\Delta t}}) + (1 - r_1 - r_2)\, \xi(x, t). \tag{13}$$

This scheme has form (8) with:

$$\begin{aligned}
D &= (1 - \alpha_1 m_1 \lambda)(1 - \alpha_2 m_2 \lambda),\\
A &= (r_1 k_1 + r_2 k_2)\lambda + (r_2 \beta_2 m_2 k_1 - r_1 \alpha_2 m_2 k_1 - r_2 \alpha_1 m_1 k_2)\, \lambda^2,\\
B &= 1 + (r_1 m_1 + r_2 m_2 - \alpha_1 m_1 - \alpha_2 m_2)\lambda\\
&\quad + (r_2 \beta_2 m_2 m_1 - r_1 \alpha_2 m_2 m_1 - r_2 \alpha_1 m_1 m_2 + \alpha_1 \alpha_2 m_1 m_2)\, \lambda^2,\\
C &= (r_1 n_1 + r_2 n_2)\lambda + (r_2 \beta_2 m_2 n_1 - r_1 \alpha_2 m_2 n_1 - r_2 \alpha_1 m_1 n_2)\, \lambda^2.
\end{aligned}$$

The term $A + B$ contains a term independent of λ. Then $A + C$ can be made proportional to λ^2 for

$$r_1 k_1 + r_2 k_2 + r_1 n_1 + r_2 n_2 = 0$$

or

$$r_1 m_1 + r_2 m_2 = 0. \tag{14}$$

Condition (14) is a necessary condition to reach stability simultaneously for positive and negative λ. Among the second order schemes which are stable, schemes in which B and C can simultaneously vanish for some value of λ, are particularly useful.

For $B = C = 0$, the amplification factor (9) is identically zero. Perturbations transferred with the characteristic velocity corresponding to this value of λ then are completely damped. A scheme of forms (11)–(13) which has this property, is the following downwind–upwind scheme:

Downwind:

$$\xi(x, t + \tilde{\Delta}t) - \xi(x, t) + \lambda(\xi(x + \Delta x, t) + \alpha\xi(x, t + \tilde{\Delta}t) - (1 + \alpha)\,\xi(x, t)) = 0. \quad (15)$$

Upwind:

$$\xi(x, t + \tilde{\tilde{\Delta}}t) - \xi(x, t) + \lambda(\alpha\xi(x, t + \tilde{\tilde{\Delta}}t) + \xi(x, t + \tilde{\Delta}t) - \alpha\xi(x, t) - \xi(x - \Delta x, t + \Delta t)) = 0. \quad (16)$$

In this scheme $m_1 = -1$, $m_2 = 1$. Condition (14) is fulfilled with $r_1 = r_2 = r/2$. Then (13) is

$$\xi(x, t + \Delta t) = 0.5r(\xi(x, t + \tilde{\Delta}t) + \xi(x, t + \tilde{\tilde{\Delta}}t)) + (1 - r)\,\xi(x, t). \quad (17)$$

Here $r > 1$ corresponds to overrelaxation. Equation (17) gives

$$\xi(x, t + \Delta t) = RX\xi(x - \Delta x, t + \Delta t) + (1 - RX^2)\,\xi(x, t) + (-RX + RX^2)\,\xi(x + \Delta x, t) \quad (18)$$

with $R = 0.5r$ and $X = \lambda/(1 + \alpha\lambda)$. Equation (18) has form (8) with

$$D = 1, \qquad A = RX, \qquad B = 1 - RX^2, \qquad C = -RX + RX^2,$$

$$B = C = 0 \qquad \text{for} \quad R = 1 \quad \text{and} \quad X = 1.$$

For $R = 1$ $(r = 2)$, stability condition (10) gives

$$(1 - \sqrt{5})/2 \leqslant X \leqslant (1 + \sqrt{5})/2; \quad (19)$$

$X = 1$ can correspond to the transfer along $u + c$. Therefore α and Δt have to be chosen according to

$$X^+ = (M + 1)C_o/(1 + \alpha(M + 1)C_o) = 1 \quad (20)$$

with $C_o = c\Delta t/\Delta x$ and $M = u/c$. The X value corresponding to other characteristic velocities should fulfill stability requirement (19).

In supersonic flow ($M > 1$), the X values corresponding to u and $u-c$, obviously are in the range $0 < X < 1$ and automatically satisfy (19). In subsonic flow ($M < 1$), the X value corresponding to $u-c$ is negative and has to fulfill

$$(1-\sqrt{5})/2 < X^-. \tag{21}$$

A combination of

$$X^- = (M-1)C_o/(1+\alpha(M-1)C_o) \tag{22}$$

and (20) gives

$$\alpha = \frac{(X^- - 1)M + X^- - 1}{2X^-}, \qquad C_o = \frac{2X^-}{1-X^-}\,\frac{1}{1-M^2}. \tag{23}$$

Both in subsonic and supersonic flow, the best scheme is reached when Δt is as large as possible. The numerical transfer velocity of perturbations transferred along u and $u-c$ is then maximised.

In subsonic flow, this is reached when X^- is as close as possible to stability limit (21). In supersonic flow this is reached when X^- is as close as possible to 1. In a practical scheme for transonic calculations, X^- can be chosen according to Fig. 1. For $M < M_o$

$$X^+ = 1, \qquad X^- = -Q \to \alpha = (M-1+Q_o)/Q_o, \qquad C_o = Q_o/(1-M^2).$$

For $M > M_o$

$$X^+ = 1, \qquad X^- = (M-1)\bigg/\left(M - M_o + \frac{1-M_o}{Q}\right) \to \alpha = (M_o - 1 + Q_o)/Q_o,$$
$$C_o = Q_o/((1-M_o)(1+M)),$$

with $Q_o = 2Q/(1+Q)$. Possible values are $Q = 0.615$ and $M_o = 0.95$.

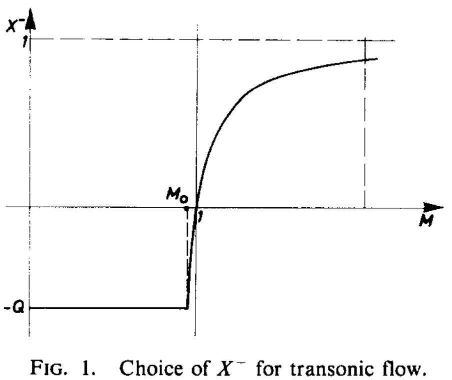

FIG. 1. Choice of X^- for transonic flow.

Application to One-Dimensional Euler Equations

The Euler equations in conservative form are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \qquad \frac{\partial \rho \mathbf{V}}{\partial t} + \nabla \cdot (\rho \mathbf{V}\mathbf{V}) + \nabla p = 0, \qquad \frac{\partial \rho E}{\partial t} + \nabla \cdot (\rho H \mathbf{V}) = 0, \tag{24}$$

The one-dimensional Euler equations are in quasi-linear form

$$(\partial \xi/\partial t) + A_1(\partial \xi/\partial x) = 0$$

with

$$\xi = \begin{pmatrix} \rho \\ u \\ v \\ w \\ p \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} u & \rho & 0 & 0 & 0 \\ 0 & u & 0 & 0 & 1/\rho \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & \gamma p & 0 & 0 & u \end{pmatrix}.$$

The discretisation according to (15), (16) is:

Downwind Step:

$$\xi(x, t + \widetilde{\Delta t}) - \xi(x, t) + A_1(\Delta t/\Delta x)(\xi(x + \Delta x, t) - (1 + \alpha)\,\xi(x, t) + \alpha \xi(x, t + \widetilde{\Delta t})) = 0. \tag{25}$$

Upwind Step:

$$\xi(x, t + \widetilde{\widetilde{\Delta t}}) - \xi(x, t) + A_1(\Delta t/\Delta x)(\xi(x, t + \widetilde{\Delta t}) + \alpha \xi(x, t + \widetilde{\widetilde{\Delta t}}) - \alpha \xi(x, t) - \xi(x - \Delta x, t + \Delta t)) = 0. \tag{26}$$

Equations (25), (26) are similar to Eqs. (15), (16) by replacing the scalar $V(\Delta t/\Delta x)$ with the matrix $A_1(\Delta t/\Delta x)$. This means that all conditions that were imposed on V have to be imposed on the eigenvalues of A_1. These are $u + c$, u, $u - c$. Perturbations transferred along $u + c$ are then completely damped. The transfer velocity corresponding to $u - c$ and u is maximised. In the acoustic movement ($u + c$ and $u - c$), waves travelling to the right are completely damped, waves travelling to the left are almost undamped but accelerated. In the convective movement, there is no damping. The attainment of the steady state is thus not determined by the internal damping but by the expulsion of perturbations along u and $u - c$. The complete damping along $u + c$ is, however, absolutely necessary to avoid reflection along $u - c$ at the outflow boundary. Since there is only higher coupling for small perturbations between the acoustic movement and the convective movement, there is asymptotically no interaction between u and $u - c$. For a sufficiently long time a perturbation along

u and $u - c$ is expulsed in a number of iterations which is proportional to the number of elements in the flow field.

The asymptotic rate of convergence is thus $O(\Delta x)$ since by increasing the number of elements in the flow field, the number of iterations necessary to reach a certain level of convergence is only linearly increased, when starting from the same initial state. The technical application of this scheme to the conservative one-dimensional Euler equations is straightforward. The nonlinear equations are linearised to form (25), (26).

The state variables at the new time level are iteratively calculated updating A_1, until the increment of the mach number per iteration becomes less than some convergence factor C. In the shock region, the usual artificial viscosity is necessary to damp the post-shock oscillations. In the momentum equations, a term is added of the form

$$D_x \Delta x(\partial^2 u/\partial x^2) \qquad \text{with} \qquad D_x = O(1).$$

The flow is calculated for a nozzle divided in 28 constant Δx segments. The section is S_o between nodes 1 and 3 and between nodes 21 and 29. Between nodes 3 and 21 the section is

$$S(i) = S_o\{0.9 + 0.1 \times (2((i-12)/9)^2 - ((i-12)/9)^4)\}.$$

The outlet pressure is $p_2 = 0.718025 \times p_{01}$. The exact solution has a shock on node 16. The calculation is done with relaxation factor $r = 2$, damping term in the artificial viscosity $D_x = 0.01$, transition mach number $M_0 = 0.95$, $Q = 0.615$, convergence factor $C = 0.015$.

Since formulas (23) are very sensitive to the mach number, M must be multiplied by a safety factor $S = 0.995$ before it is used in (23). Overestimation of M causes X^-

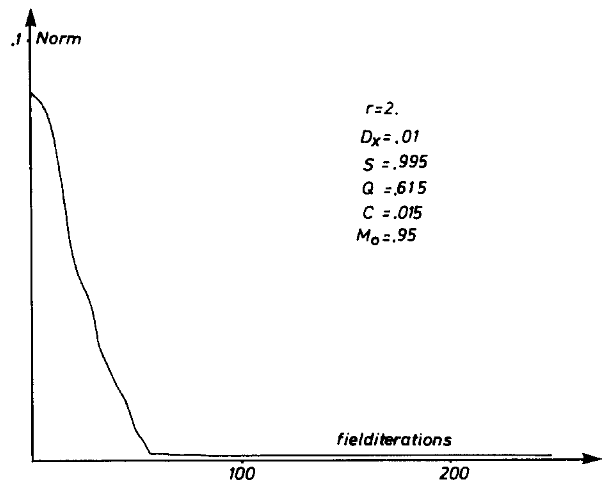

FIG. 2. Convergence history for the one-dimensional calculation with 28 elements.

TABLE I

One-Dimensional Calculations for $D_x = 0.01$, $S = 0.995$, $Q = 0.615$, $C = 0.015$, $M_o = 0.95$

Number of elements	28	56	84	112
Optimal relaxation factor	2.000	2.000	1.996	1.992
Field iterations for an accuracy 0.01	55	104	153	201
Idem for 0.001	—	119	168	218
Nodal iterations for an accuracy 0.01	1828	6428	13382	23173
Idem for 0.001	—	7262	14637	25073
Final accuracy	0.00201	0.00053	0.00044	0.00027

to exceed the stability limit $X^- = -0.618$. A convergence norm is calculated as the mean absolute deviation in the nodal points between the calculated mach number and the exact mach number. The initial state is a uniform flow with $M = 0.6$. The convergence history for the norm is depicted in Fig. 2. The steady state is reached very abruptly. This is due to the expulsion mechanism on which the convergence is based. The calculations are repeated for the same geometry but subdivided in 56, 84, and 112 elements. The norm is still calculated in the nodal points of the first case. The accuracy is defined as the value of the norm. The results are shown in Table I. These results clearly show the linear convergence rate.

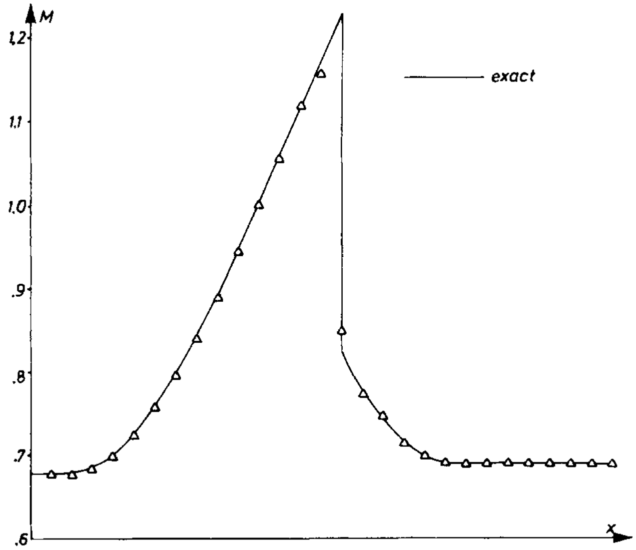

FIG. 3. Mach number distribution for the one-dimensional calculation with 28 elements.

Figure 3 contains the mach number distribution for the coarsest grid. The algorithm has 6 parameters. Four of them, relaxation factor r, safety factor S, largest negative X, and transition mach number M_0 have a universal value. The convergence rate increases when these parameters approach their theoretical values $r = 2$, $S = 1$, $X^- = -Q = -0.61803$, $M_0 = 1$. These values cannot be reached for stability reasons. Practical values are $r = 1.99$, $S = 0.995$, $Q = 0.615$, $M_0 = 0.95$. The convergence factor C and the damping factor D_x have a problem dependent value.

The convergence factor should be low enough to ensure sufficient convergence in a nodal calculation. Insufficient convergence in a nodal calculation causes instability. A convergence factor that is too safe, however, augments the computational effort without influencing the convergence rate. A rule of thumb is that after a number of field iterations equal to the number of elements in the field, only one iteration is done per nodal calculation. The damping factor D_x has no influence on the convergence rate, but has a detrimental effect on the accuracy. It should be kept as low as possible. The necessary value is strongly dependent on the shock strength.

Two-Dimensional Applications

The one-dimensional scheme can immediately be extended to two-dimensional applications for line relaxation on a mesh in which the transversal lines are straight and parallel. In formulas (23) the mach number is to be replaced by the component of the mach number in the direction perpendicular to the transversal lines.

The discretisation can easily be done by the finite volume technique. The upwind and the downwind volume corresponding to the mesh point (i, j) are depicted in Fig. 4. In order to have formal similarity with the one-dimensional algorithm, the time derivative term has to be distributed on the points $(i+1, j)$, (i, j), $(i-1, j)$ proportional to the coefficients of these points for the flux through AB.

The stability analysis on this two-dimensional discretisation reveals that there is a slight instability in the transversal direction. This instability can easily be compen-

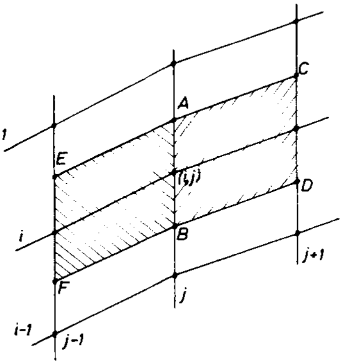

FIG. 4. Upwind and downwind volumes.

sated by a small artificial viscosity in this direction. It can, for instance, be done by introducing in the momentum equations a corrected viscosity of the form (for the X-momentum)

$$D_{y1}\,\Delta y(\partial^2 u/\partial y^2)(t^*) - D_{y2}\,\Delta y(\partial^2 u/\partial y^2)(t_o).$$

Here t^* is taken to be $t+\widetilde{\Delta t}$ in the downwind step and $t+\widetilde{\widetilde{\Delta t}}$ in the upwind step. The term t_o is a time level that is fixed for a cycle of N_v iterations and renewed after that cycle. Then $D_{y1} - D_{y2}$ is sufficiently small. Typical values are $D_{y1} = 1$, $D_{y2} = 0.99$. With these values the cycle should have approximately as many iterations as there are nodes in the transversal direction of the flow field, in order to ensure stability. This viscosity mechanism has no influence at all on the convergence rate since the convergence is driven by the expulsion mechanism in the longitudinal direction and not by the damping mechanism in the transversal direction.

The same nozzle as was used in the one-dimensional example is subdivided into 4×28 and 8×56 elements with constant height on a transversal line. The calculation is also done for a discretisation with 8×28 and 4×56 elements. In the first case the width of the nozzle is doubled, in the second case the length is doubled. This guarantees that the form of the elements is similar in all calculations. As a consequence the optimal values of the problem dependent parameters are equal. These parameters are $r = 1.99$, $S = 0.995$, $Q = 0.615$, $M_o = 0.95$, $D_x = 0.01$, $D_{y1} = 1$, $D_{y2} = 0.99$, $C = 0.013$. Let $N_v = 5$ when there are 5 nodes in the transversal direction. Let $N_v = 9$ when there are 9 nodes.

The initial state is a uniform flow with $M = 0.6$, the outflow pressure is $0.718025\times$ inlet total pressure. The results are shown in Table II. An equivalent field

TABLE II

Two-Dimensional Calculations

Number of elements	4×28	8×28	4×56	8×56
Final value of the mean mach number	0.78444	0.78375	0.78860	0.78868
Number of field iterations for a mean mach number = $0.999\times$ final value	57	58	132	135
Corresponding line field iterations	1742	1764	7483	7207
Corresponding equivalent field iterations	64.5	65.3	136.0	131.0
Final value T_{02}/T_{01}	0.99938	0.99987	0.99954	0.99958
Final value $\dot{m}_2/\dot{m}_1$	0.99950	0.99945	0.99953	0.99993

iteration is defined as 27 or 55 line iterations. These results clearly show the full conservativity of the used finite volume method and the linear convergence rate in Δx. They prove that the number of nodes in the transversal direction has no influence at all on the convergence rate.

Conclusions

By the preceding theory and computational examples, it is proved that the relaxation method can be extended to Euler equations. This gives a method with convergence rate $O(\Delta x)$. Practical experience reveals that the gain in computational efficiency in comparison with time marching techniques with convergence rate $O(\Delta x^2)$ is in the order of a factor 6 on a typical 1000 mesh point grid.

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